Topological Defects in Physics

Essay for the "Differential Geometry in Physics" course

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Abstract

This essay describes the basics of homotopy theory, its relation to group theory and its use in several branches of modern physics for characterising topological defects. The goal of the essay is to show that this is important in condensed matter physics, cosmology and elementary particle physics, branches of physics not usually related, and to provide a succinct and intuitive introduction to the subject and understanding of the phenomena. At least basic knowledge of (Lie) group theory is assumed and basic understanding of quantum field theory is necessary to follow examples from cosmology and elementary particle physics.

Contents

1 Introduction

Whenever we inspect a physical quantity in space (or spacetime) we first have to determine the set of values it can acquire at every point. Let's call this the *parameter space*[1] R and let's assume the parameter space to be a manifold. Probably the most typical example is that of spins in matter. If we limit spins to a plane then we can represent them by unit vectors in a plane and their parameter manifold is S^1, a circle, and if we allow them to point anywhere then their parameter manifold is S^2, a sphere.

In physics one is usually guided by criteria like minimization of energy and maximization of entropy to find stable configurations. But a question arises if some topological features of the physical and/or parameter space (dimensionality being the most basic one) introduce additional criteria that have to be met by a configuration, possibly preventing it from relax-

[1] In condensed matter physics the physical quantity would be called an *order parameter* and the set of values the *order parameter space*, but in general it can be something more abstract like a wave function in which case a more general terminology such as *manifold of internal states* would be fitting, but also in my opinion too clumsy and vague for our purposes.

ing to the globally most favored one.
Topological defects are parts of the domain where the quantity of interest is not necessarily everywhere well defined and the configuration is unable to satisfy the usual physical criteria (e.g. minimal energy) because to attain it would cost infinite (or the practical equivalent) energy. The reason for the infinite cost in energy is that no local "surgery" (rearrangement of the physical quantity) would suffice but a change would have to propagate infinitely far. The existence and some limitations to the behaviour of such defects, considering only the topology of the physical and parameter spaces, is best studied by a purely mathematical theory called *homotopy theory*.
Considering these special, in a way non-optimal, configurations of a physical parameter the questions of their creation and dynamics also arise. Some of the answers to the question of creation can be found in phase transitions and symmetry breaking, usually related to very fundamental problems, but for less energetic and more common ones creation can result from simpler things like external forces. Questions of dynamics are mostly answered by non-topological, physical considerations but some, like defects meeting, coalescing or crossing, are partially answered by homotopy theory itself, which we now move on to.

2 Homotopy theory

The basic idea of homotopy theory is to use *continuous deformation* of objects in a topological space to establish their equivalence. Topologically different spaces will then have different classes of such equivalent objects. Since members from different classes of those objects cannot be deformed into one another that can also tell us something about physics if we map a physical quantity to such objects.
The simplest kind of a deformable object is a path, or its closed form - a loop. Mathematically, a path on a topological space X is a map $\gamma : I \to X$ where $I = [0, 1]$, and a loop (at x_0) is a path with $\gamma(0) = \gamma(1) = x_0 \in X$, or put simply it's end points being the same. It is clear that a loop is essentially a circle, S^1. This definition can be easily generalized to n dimensions by making a mapping $\gamma^n : I^n \to X$ where $I^n = I \times \ldots \times I$ and identifying all boundary points as the same point, i.e. $s = x_0, \forall s \in \partial I^n$ where $x_0 \in X$ and $\partial I^n = \{(s_1, \ldots, s_n) \in I^n |$ at least one $s_i = 0$ or $1\}$. It should also be clear that such an n-loop is essentially an n-sphere, S^n.
The basis of homotopy theory is the "*homotopic to*" relation between two n-loops α and β at x_0 which is formally defined as an existence of a continuous map $F : I \times I^n \to X$ such that:

$$\begin{aligned} i) \quad & F_0(s) = \alpha(s), \quad \forall s \in I^n \\ ii) \quad & F_1(s) = \beta(s), \quad \forall s \in I^n \\ iii) \quad & F_t(0) = F_t(1) = x_0, \quad \forall t \in I \end{aligned} \tag{2.1}$$

The crucial thing is that the map be continuous in which case it essentially represents a continuous deformation of α to β, as is depicted in figure 1[2]. Such a map is called *a homotopy*.

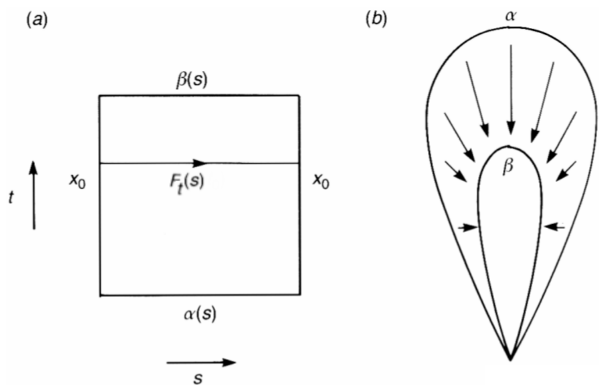

Figure 1: (a) representation of the formal definition, (b) the idea of a continuous deformation

The most important thing to notice is that the defined relation is an equivalence relation since a loop is homotopic to itself, the relation is clearly symmetric and if there exists a homotopy between α and β and β and γ it's trivial to construct a homotopy between α and γ. Being an equivalence relation makes "*homotopic to*" separate (n-)loops naturally into equivalence classes of loops that are topologically the same in an intuitive sense[3].
With loops defined in such a way it is easy to define operations like the product and the inverse, which are

[2] This figure is taken from [1] where you can find the basics of homotopy and homology theory presented a bit more formally, more in depth and with more instructive figures. I also recommend [2] for the same purpose.

[3] As is later commented, this is really the property of freely homotopic loops and thus conjugacy classes of the equivalence classes in the framework of homotopy groups. This distinction is, however, only relevant in noncommutative homotopy groups.

needed to construct algebraic structures like groups. The product of two (n-)loops is naturally defined by just "gluing" them together, first going around one and then around the other. Formally it is done by reparametrisation, going twice as "faster" around each one:

$$(\alpha * \beta)(s) = \begin{cases} \alpha(2s) & 0 \leq s \leq \frac{1}{2} \\ \beta(2s-1) & \frac{1}{2} \leq s \leq 1 \end{cases} \tag{2.2}$$

The inverse of a loop is defined by going the other way around it:

$$\alpha^{-1}(s) := \alpha(1-s) \tag{2.3}$$

ensuring that a product of a loop with its inverse doesn't "surround" anything, i.e. is always homotopic to a constant path (a point).

2.1 Homotopy (fundamental) groups

The equivalence classes of homotopic loops, using the defined product and inverse, form a group structure. To prove it formally we need to define a product of equivalence classes and an inverse of an equivalence class and then show that the three conditions for a group hold: associativity, the existence of a unit element and an inverse element that in a product with the original gives the unit element. Formal proofs can be found in [1] and [2] but intuitively one only needs to realise that:

a) a product of two loops α and β is homotopic to all the loops that can be made by a product of any two loops homotopic to α and β, i.e. $[\alpha * \beta] = [\alpha] * [\beta]$, because if you can deform a loop to α and another to β then you can also deform their product to $\alpha * \beta$, and

b) a product of a loop and its inverse is not itself a point but it is homotopic to a point, i.e. $[\alpha * \alpha^{-1}] = [x_0]$.

From these two realisations it should be easy to see why the equivalence classes of loops $[\alpha]$ form a group. That group is called the *fundamental group* or the *first homotopy group* $\pi_1(X)$ if we are using regular 1D loops and the *nth homotopy group* $\pi_n(X)$ if we are using n-loops. For the sake of completeness it should be said that homotopy groups are topological invariants meaning that two homeomorphic topological spaces must have the same homotopy groups or put in other words, a homeomorphism preserves the homotopy groups. Also, an important feature of the homotopy groups is that the homotopy group of a direct product of topological spaces is the direct sum of their respective homotopy groups:

$$\pi_n(X \times Y) \simeq \pi_n(X) \oplus \pi_n(Y) \tag{2.4}$$

To return to physics for a second, the basic idea of the application of homotopy theory to physics is to describe defects by looking at the values of a physical parameter on an n-loop surrounding a defect and classifying it by the corresponding element of the nth homotopy group.
One of the nuances I've left out in the preceding "derivation" of homotopy groups is that I've been using loops *at a point*, anchored at some x_0, all the time, but in the definition of the homotopy groups there is no reference to any particular point. The reason is that there always exists a natural isomorphism between homotopy groups at any two points of a path-connected manifold. Another nuance is that the isomorphisms don't have to be unique and that is related to the fact that *freely homotopic*[4] loops are not necessarily homotopic at a point, which is nicely seen in figure 2. The

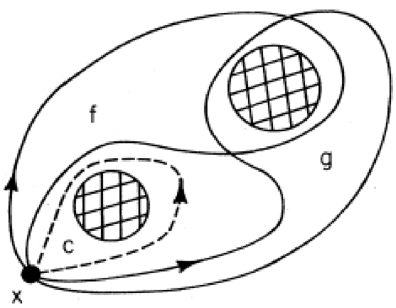

Figure 2: Two loops f and g that are freely homotopic but not homotopic at x

relation between freely homotopic and based homotopic loops is relatively simple with 1D loops where it is directly related to the commutativity of $\pi_1(X)$ and its conjugacy classes. I will elaborate on this a bit

[4]The definition of freely homotopic being the natural one of continuous deformation without being anchored at any one point.

because it has to do with the physical consequences of topological defects meeting, crossing or interacting in some other way.

If $\pi_1(X)$ is commutative then its conjugacy classes are made up of single elements and loops that are homotopic are also freely homotopic. In that case topological defects like monopoles can combine into a single topological defect without any ambiguity and dependence on the way they are combined, and topological defects like strings upon crossing just pass through each other without entangling. If, however, $\pi_1(X)$ is noncommutative then its conjugacy classes are not trivial and loops are freely homotopic if they belong in the same conjugacy class, meaning even if they are not homotopic at a point. The combination of topological defects is then determined by the product of conjugacy classes which can give an ambiguous answer to the question of the conjugacy class of the resulting topological defect and in fact the conjugacy class of the resulting defect can depend on the path by which the two defects were joined, relative to other defects in the medium. The physical consequence of this is that some defects can in some cases be "undone" by things like splitting and combining after traversing a certain path around some other defects (like in the biaxial nematics example given later). In the case of crossing string defects the noncommutativity of the homotopy group results in their entangling if the defects belong to elements of the homotopy group that do not commute. The physical consequences are that a new string defect is created connecting the two and, since a defect carries some energy, prevents the two original defects from separating. Things like this would be very hard to understand and predict without the machinery of topology and homotopy theory.

A more detailed exposition of the nuances of isomorphisms, noncommutativity and free homotopicity I mentioned here is given in [3] using a very intuitive approach and a lot of insightful figures[5].

2.1.1 Higher homotopy groups

Most of what has been presented so far is applicable not only to the fundamental group but also to the higher homotopy groups, the only difference being that the objects being deformed are n-loops, or equivalently n-spheres.

One of the biggest differences is that higher homotopy groups are always commutative, but that doesn't save them (or us) from the complications of the difference between homotopic and freely homotopic n-loops and the resulting complications in the interactions of topological defects described by those higher homotopy groups. The reason is that the said difference fundamentally has to do with the uniqueness of the isomorphisms between homotopy groups at different points, and the connection between points is always made by paths and is thus always related to $\pi_1(X)$. More on that can also be found in [3] (sections III. B. and IX. A.).

Higher homotopy groups are needed because topological defects can be described by homotopy groups of loops that can surround those defects. So for example, a monopole in 3 spatial dimensions has to be described by $\pi_2(X)$ because it can only be surrounded by a 2-sphere.

2.2 Homotopy on group manifolds

The determination of homotopy groups is made much simpler if the topological space one is dealing with is a group manifold, and even more so a simply connected group manifold. The reason for that is a series of theorems that I will state without proof or much explanation, one or both of which can be found in the already mentioned review by Mermin [3].

The first very useful theorem says that for a simply connected, continuous group G and any subgroup H of G the fundamental group of the coset space $\pi_1(G/H)$ is isomorphic to the quotient group H/H_0, where H_0 is the connected component of H containing the identity, or put simply:

$$\pi_1(G/H) \simeq H/H_0 \tag{2.5}$$

So the "only" thing one needs to do to find the fundamental group of a parameter space is to find a simply connected group and its subgroup whose coset is isomorphic to the parameter space. This sounds completely abstract and unpractical but is actually completely opposite and can often be done using simple symmetry arguments.

[5] This review by N. D. Mermin gives a great introduction to homotopy theory and topological defects in ordered media and much of my understanding of the subject and the contents of this essay is derived from it.

The second theorem concerns the second homotopy group and states simply that if $\pi_2(G) = \{e\}$ then:

$$\pi_2(G/H) \simeq \pi_1(H) \tag{2.6}$$

Both stated theorems can be extracted from a more general theorem that says that the sequence of homomorphisms:

$$\begin{aligned} \ldots \to \pi_n(G) \to \pi_n(G/H) \to \\ \to \pi_{n-1}(H) \to \pi_{n-1}(G) \to \ldots \end{aligned} \tag{2.7}$$

is a thing called an exact sequence. In such a sequence it holds that if in $G_1 \to G_2 \to G_3 \to G_4$ the groups G_1 and G_4 are trivial (have only the identity) then the homomorphism between G_2 and G_3 is an isomorphism. Applied to (2.7) it tells us that if $\pi_n(G) = \pi_{n-1}(G) = \{e\}$ then:

$$\pi_n(G/H) \simeq \pi_{n-1}(H) \tag{2.8}$$

As I already pointed out, it seems rather abstract and unpractical to formulate a parameter space as a simply connected group or a coset of groups. But actually it is the most natural thing because a parameter space is directly related to the reduction of symmetry of the physical parameter in relation to the symmetry of the whole physical space. For example, a plane has $SO(2)$ as its symmetry group. The isotropy subgroup of $SO(2)$ for a planar spin is just $\{e\}$ since any other rotation changes the planar spin. Thus, the parameter space for a planar spin is isomorphic to $SO(2)/\{e\} = SO(2) \simeq S^1$. This was a trivial example. A more complex and persuasive example is the spin in three dimensions. The whole space has $SO(3)$ as its symmetry group, and the isotropy subgroup of $SO(3)$ for a spin is $SO(2)$ because all rotations about the axis of the spin leave it unchanged. Thus, the parameter space for a spin is isomorphic to $SO(3)/SO(2)$ which is known to be $\simeq S^2$.

So basically the only practical obstacle for using the theorems given in this section for easy calculation of homotopy groups are the conditions on G being n-connected (meaning $\pi_n(G) = \{e\}$) because they are generally hard to satisfy ($SO(3)$ is not even simply connected). However, the condition for simple connectedness is relatively easy to achieve by taking the universal covering group of the natural symmetry group G since it always exists and is simply connected. An additional convenience is then given by a theorem stating that for $n \geq 2$ the nth homotopy groups of a group G and its covering group $\tilde{G}$ are isomorphic. But even if one manages to find the universal covering group, which is very often tricky, additional complications arise from the need to "lift" the isotropy subgroup H to the appropriate subgroup of the universal covering group.

A lot more formal and mathematical description of defects and symmetry with advanced usage of group theory is given in [4]. The following examples will show how symmetry groups, group theory and homotopy theory are used together to predict and classify topological defects and give constraints on their dynamics.

3 Application in ordered media

The domain of condensed matter physics and ordered media is the natural intuitive area to study defects and apply homotopy theory since the order parameters are physical quantities one can easily imagine by vectors, rods and similar familiar objects. We will mostly be concerned here by continuous media although media with broken translational symmetry, those possessing a lattice, have an abundance of defects that are of a topological character. Typical of those are dislocations and disclinations (figure 3). The reason why we will not deal mathematically with media with broken translational symmetry here is because the mathematics is even more complicated and unclear. More on that can be found in [3] (section VIII.).

As already mentioned in the introduction, generation of defects in continuous media can happen in phase transitions but also by simple influence of external forces like bending of a metal bar. Entanglement of line defects and other interactions, mentioned earlier with noncommutativity, can in principle have physical consequences although no materials having that property have been found so far that I am aware of. A simple introduction to and overview of the classification of materials and topological defects in them, in just a few pages, is given in [5].

Topological defects in materials are usually divided

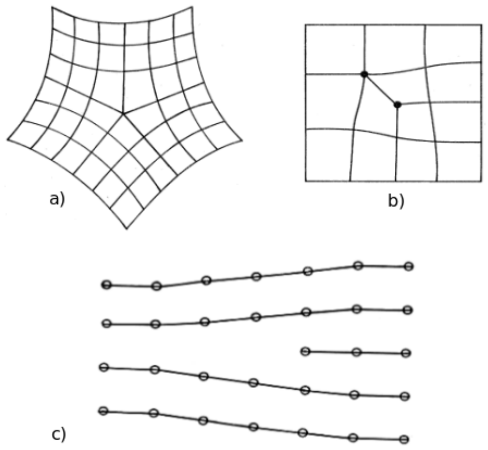

Figure 3: (a) a $-90°$ disclination, (b) two disclinations ($-90°$ and $+90°$) making a (1,1) dislocation in the far configuration, (c) a (1,0) dislocation

into:

a) **monopoles** - point defects in 3D described by $\pi_2(R)$.

b) **vortices** - point defects in 2D or, equivalently, line defects in 3D, both described by $\pi_1(R)$.

c) **domain walls** - plane defects in 3D, but also lines in 2D or points in 1D. A structure that separates two distinct parts of the domain and is thus described by $\pi_0(R)$.

d) **textures** - those are solutions that are not trivial, uniform configurations and cannot be transformed into one with finite "surgery", but are not strictly defects. They are also sometimes called *solitons* and are described by $\pi_3(R)$.

What defines a topological defect is that it has influence in the medium far away from the defect itself, in other words it can be detected non-locally. This is plain to see from the use of homotopy theory because the size of the n-loop doesn't matter, only the class it belongs to, so one can detect or determine a defect in a configuration by taking an arbitrarily large n-loop, even "at infinity". This is the reason why textures are formally not defects. They are defined such that the parameter takes on the same value everywhere at infinity, which is why the $\mathbb{R}^3$ space is practically compactified to S^3. A configuration as a whole can then be classified by $\pi_3(R)$ by making the whole of physical space play the role of the 3-loop from which a mapping is made to the parameter space R. That configuration can be non-trivial but it cannot be said that it possess a defect, more that it is a "defect" by itself. It is instead said that the configuration has a certain texture.

The following sections will present typical order parameters in condensed matter physics and analyze what kind of topological defects they admit.

3.1 Spins

Perhaps the simplest model of a parameter space is given by planar spins, which were already mentioned in the introduction. In this model the spin is represented by a vector confined to a plane and can thus be represented by a single angle ϕ, the mapping from physical space to parameter space being:

$$f(\vec{r}) = \cos\left(\phi(\vec{r})\right)\hat{u} + \sin\left(\phi(\vec{r})\right)\hat{v} \quad .$$

It is then obvious that the parameter space is isomorphic to S^1 and also fairly straightforward to see that $\pi_1(S^1) \simeq \mathbb{Z}$ because you can "wind" a loop around a circle as many times as you like in any direction and loops with different "winding numbers" would not be possible to continuously deform into each other (figure 4). But what about higher homotopy groups? To avoid relying on intuition and being able to easily answer further questions is the reason to use group manifolds and the associated theorems presented in section 2.2.

In the case of planar spins the natural group manifold G is $SO(2)$ and the isotropy subgroup H is trivial. The universal covering group $\tilde{G}$ of $SO(2)$ is the 1D translation group $T(1)$ which is isomorphic to $\mathbb{R}$ but then the isotropy subgroup is lifted to:

$$\tilde{H} = \{2\pi k \mid k \in \mathbb{Z}\}$$

because all real numbers $\phi + 2\pi k$ correspond to the same element of $SO(2)$. Using theorem (2.5) then gives:

$$\pi_1(SO(2)) \simeq \pi_1(T(1)/\tilde{H}) \simeq \tilde{H} \simeq \mathbb{Z}, \qquad (3.1)$$

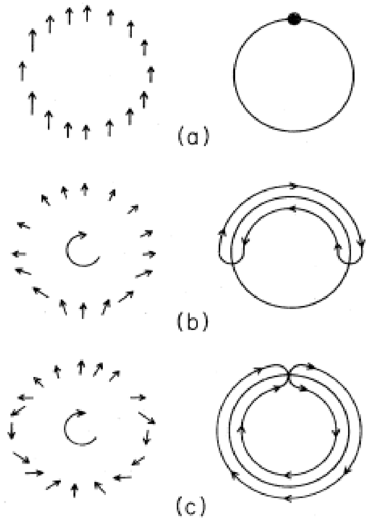

Figure 4: (a) uniform configuration, (b) a configuration that can be brought to a uniform one, meaning winding number 0 and no defects, (c) a defect with winding number +2

Also the use of (2.8) simply gives the general result:

$$\pi_n(SO(2)) = \{e\} \tag{3.2}$$

for $n \geq 2$ since $\pi_{n-1}(\mathbb{Z}) = \{e\}$ because it is discrete. So we conclude that planar spins can make only vortices, that the vortices can be labeled by integers and that, since $\mathbb{Z}$ is commutative, two or more vortices combine into a vortex that is labeled simply by the sum of their labels (figure 5).

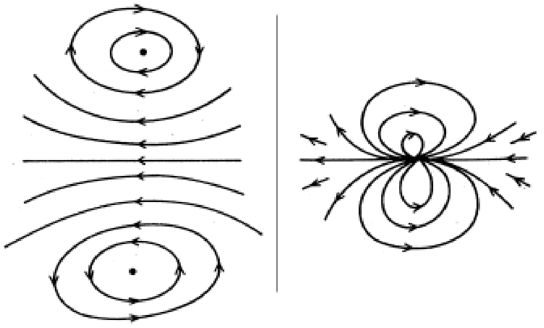

Figure 5: Two +1 defects (left) joining into a single +2 defect (right).

For the also already mentioned case of ordinary 3D spins one can also get some results by intuition. For example, since the parameter space is S^2, a sphere, it is obvious that $\pi_1(S^2) = \{e\}$ because everybody knows that you cannot "lasso a sphere". Similarly one can guess that $\pi_2(S^2) = \mathbb{Z}$ from the analogy with $\pi_1(S^1)$. But let us use the group-theoretic approach. As was already explained in section 2.2, the G for ordinary spins is $SO(3)$ and the isotropy subgroup H is $SO(2)$. The universal covering group of $SO(3)$ is the group of unitary 2×2 matrices with positive unit determinant - $SU(2)$, which is commonly found in physics and the reader may know it is a double cover of $SO(3)$. The appropriate lift of $SO(2)$ is then simply to $U(1)$ and since $U(1)$ is connected it immediately follows from (2.5) that:

$$\pi_1(S^2) = \pi_1(SU(2)/U(1)) = U(1)/U(1) = \{e\}. \tag{3.3}$$

Since $SU(2)$ is 2-connected (it can be represented as a ball) it also follows simply from (2.6) that:

$$\pi_2(S^2) = \pi_2(SU(2)/U(1)) = \pi_1(U(1)) = \mathbb{Z}, \tag{3.4}$$

as expected. We cannot say much about higher homotopy groups so easily because the condition for the use of (2.8) is not satisfied, that is $\pi_n(SU(2)) \neq \{e\}$ for $n > 2$. That the use of (2.8) would be erroneous is seen from the fact that using it would give the result: $\pi_3(S^2) = \pi_2(U(1)) = \{e\}$ when in fact the correct result is:

$$\pi_3(S^2) = \mathbb{Z} \tag{3.5}$$

To summarize, ordinary spins cannot make vortices but can make monopoles whose nature is the same as those of vortices for planar spins. Additionally, there also exists the possibility of textures, solitonic configurations that are in themselves not defects but prevent the material from relaxing to the uniform configuration.

3.2 Nematics

Nematics are liquid crystals in the nematic phase which is the most common of thermotropic liquid crystal phases. Other phases include smectic and cholesteric both of which have interesting but more complicated topological features (see table 2 in [4]).
Nematics are basically consisted of rod-shaped molecules that like to align. Their interesting topological features come from the fact that the orientation of the molecules doesn't matter, which enables us to represent them by headless vectors called directors. The

"only" thing different from ordinary spins is that the isotropy subgroup includes rotations by π around axes perpendicular to the director axis. This makes the parameter space $SO(3)/D_\infty \simeq RP^2$ which is the real projective plane and corresponds to a 2-sphere with opposite points identified. This matches the intuitive definition of a director as a headless vector.
In order to use $SU(2)$ instead of $SO(3)$ we need to find the corresponding lift of D_∞. It is made of two sets of elements connected by a π rotation around an axis perpendicular to the director axis, for example $g = u(\hat{x}, \pi)$:

$$u(\hat{z}, \theta) = H_0 \simeq SO(2)$$
$$u(\hat{x}, \pi) u(\hat{z}, \theta) = gH_0$$

This results in:

$$\pi_1(SU(2)/H) \simeq H/H_0 \simeq Z_2 = \{0, 1\}. \quad (3.6)$$

which is a known result for $\pi_1(RP^2)$. The consequence of this is that nematics can form vortices, but only of one kind (figure 6) and whenever two of them meet they annihilate.
Since S^2 is the covering group of RP^2 their higher homotopy groups are the same according to a theorem mentioned at the end of section 2.2. But this doesn't mean that the conclusions about monopoles and textures in ordinary spins can be completely copied to nematics. The reason for caution is stated in section 2.1.1 and can intuitively be related with the isotropy subgroup H having more than one connected component. So even though the higher homotopy groups of H are determined completely by H_0, the law of combination of different topological defects is not. One of the consequences for nematics is that monopoles with labels $+n$ and $-n$ are topologically the same which leads to the unusual possibility of a monopole with an even label $2n$ separating into two equal n monopoles and combining back into a defectless configuration.

Another interesting material is the **biaxial nematic** which aligns by two axes. It can be imagined as elongated rectangles which are unchanged only by π rotations around the three perpendicular axes, i.e. the four-element point group D_2. It's parameter space is then $R \simeq SO(3)/D_2$ but to apply the theorems from section 2.2 we need to use $SU(2)$ and the equivalent lift of D_2 which happens to be the quaternion group Q (more details and intuition in [3]). The quaternion group is a discrete group consisted of 8 elements: the unit element, its negative, 3 "imaginary" units i_1, i_2 and i_3, and their negatives. The imaginary units are connected by the totally antisymmetric Levi-Civita tensor via $i_a i_b = \epsilon_{abc} i_c$. The result is that, by the theorems of section 2.2, the first and second homotopy groups for biaxial nematics are:

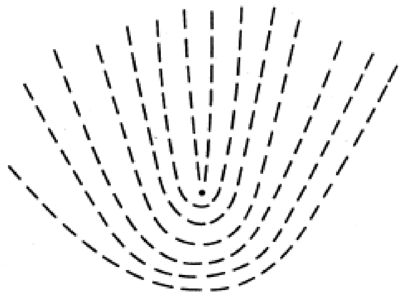

Figure 6: A vortex in a nematic

$$\pi_1(R) = \pi_1(SU(2)/Q) = Q \quad (3.7)$$

$$\pi_2(R) = \pi_1(Q) = \{e\} \quad (3.8)$$

This tells us that monopoles are unable to form, unlike in normal nematics, and that vortices are determined by a very special and unexpected group.
What is especially interesting is that Q is obviously noncommutative so the 8 elements have to be grouped into conjugacy classes to give topologically distinct vortices. There are five of those and they define four different kinds of defects, those that are 2π rotations and those that are π rotations around the three distinct axes. The law for the combination of those defects is very peculiar and gives the possibility of a defect catalyzing the destruction of another defect. Again, more details can be found in [3] but the point to be taken is that quite unusual topological defects appear in real life and it would be very hard, if not impossible, to understand them, let alone predict their existence and behaviour, without the use of homotopy theory.

3.3 Superfluid helium-3

Although condensed matter can be very complicated and exhibit some very peculiar properties, the discovery and subsequent theoretical explanation (BCS theory etc.) of superfluidity and superconductivity made it especially hard, probably impossible, for condensed matter physicists to use their intuition and previous experience in predicting and explaining their behaviour. Both phenomena are described in the framework of quantum field theory but in a very different way and context then the examples of the next section. The consequence is that the order parameters are unlike anything used so far, completely unintuitive and mathematically abstract. Two of them will be presented here but without any physical motivation or explanation because the theories are too complicated.

Superfluid helium is basically the trigger that brought topologists and homotopy theory to condensed matter physics because condensed matter physicists couldn't cope with it using conventional methods they used and developed until then. Two isotopes of helium are superfluid, ^3He and ^4He, but the more interesting and complicated one is helium-3 which is theoretically very similar to superconductors since both effects occur with fermions. Helium-3 has two superfluid phases, the A and B phase, and each has various regimes that are described by different order parameters. We will deal here with the dipole-locked and the dipole-free A phase.

The simpler of the two is the **dipole-locked A phase** whose order parameter is a complex vector field:

$$\boldsymbol{e}(\vec{r}) = \hat{\phi}_1(\vec{r}) + \imath\hat{\phi}_2(\vec{r}) \qquad (3.9)$$

with the constraints: $|\boldsymbol{e}|^2 = 1$ and $\boldsymbol{e} \cdot \boldsymbol{e} = 0$.

It can be imagined as a pair of orthonormal axes $(\hat{\phi}_1, \hat{\phi}_2)$ at each point of space and it is then easy to see that its parameter space R can be described by the full group manifold of $SO(3)$ since no element of $SO(3)$ but the identity leaves it unchanged. As usual, we use $SU(2)$ instead of $SO(3)$. The lift makes the isotropy group become non trivial since the identity in $SO(3)$ corresponds to both the identity and its negative in $SU(2)$. The theorems of section 2.2 then give:

$$\pi_1(R) = \pi_1(SU(2)/\{e, -e\}) = Z_2 \qquad (3.10)$$

$$\pi_2(R) = \pi_1(\{e, -e\}) = \{e\} \qquad (3.11)$$

This seems like the poorest homotopy structure so far; it allows only one kind of vortices, like the nematics, and no monopoles. However, using the facts that $SO(3) \simeq RP^3$, that S^3 is the covering space of RP^3 and that higher homotopy groups of a space and its covering space are the same we can conclude that:

$$\pi_3(R) \simeq \pi_3(S^3) = \mathbb{Z} \qquad (3.12)$$

which means that the dipole-locked A phase of helium-3 admits an infinite variety of textures. Examples of such textures are the Anderson-Toulouse and Mermin-Ho vortices as well as the Shankar monopole (see [1], section 4.10).

The **dipole-free A phase** has an order parameter that doesn't seem a lot more complicated but results in a much richer and complicated topology. The order parameter is defined as:

$$A(\vec{r}) = \hat{n}(\vec{r})\boldsymbol{e}(\vec{r}) \qquad (3.13)$$

where $\boldsymbol{e}$ is the same as in (3.9) and $\hat{n}$ is an uncoupled unit vector.

The natural first choice for the parameter space would be just the direct product of the parameter spaces for each of the separate components, which is: $(SO(3)/SO(2)) \times SO(3)$, or equivalently $SO(3) \times SO(3)$ with the isotropy subgroup $SO(2) \times 1$. But care must be taken of the fact that if both components simultaneously switch signs it amounts to no change at all. So alongside $(R(\hat{z}, \theta), 1)$ transformations of the form $(R(\hat{u}, \pi), R(\hat{z}, \pi)) = (R(\hat{x}, \pi)R(\hat{z}, \theta), R(\hat{z}, \pi))$ also belong to the isotropy subgroup. To understand the elements of the isotropy group one can imagine the $SO(3) \times SO(3)$ space as two coordinate systems where the first is aligned with the $\hat{z}$-axis pointing in the $\hat{n}$ direction and the second aligned such that the $\hat{z}$-axis is perpendicular to the $(\hat{\phi}_1, \hat{\phi}_2)$ plane. The direction of $\hat{u}$ is perpendicular to the $\hat{z}$-axis and makes an angle θ with the $\hat{x}$-axis.

To use the theorems of section 2.2 we need to use $SU(2) \times SU(2)$ as the group manifold G and lift the isotropy subgroup accordingly. This results in four sets of isotropy transformations that can be related to an element $g = (u(\hat{x}, \pi), u(\hat{z}, \pi))$ as left cosets:

$$(u(\hat{z},\theta),1) = H_0 \simeq SO(2)$$
$$(u(\hat{z},\theta),-1) = g^2 H_0$$
$$(u(\hat{x},\pi)u(\hat{z},\theta), u(\hat{z},\pi)) = gH_0$$
$$(u(\hat{x},\pi)u(\hat{z},\theta), -u(\hat{z},\pi)) = g^3 H_0$$

Using theorems (2.5) and (2.6) then leads us to the following results:

$$\pi_1(G/H) \simeq H/H_0 \simeq Z_4 \tag{3.14}$$
$$\pi_2(G/H) \simeq \pi_1(H) \simeq \pi_1(SO(2)) \simeq \mathbb{Z}. \tag{3.15}$$

This leads to the conclusion that there can be three different types of vortices in the dipole-free A phase of helium-3 and that they are nicely behaved, as far as topology is concerned. But, as with nematics, one should not be quick to draw conclusions on the behaviour of monopoles and textures since the isotropy subgroup contains more than one connected component. In fact, monopoles have a similar property to the ones in nematics (more on that can be found in [3]).

4 Application in quantum field theory

In quantum field theory the quantum field ϕ is described by a Lagrangian density from which the field dynamics equations and the energy of the field can be derived. The Lagrangian always contains a potential $V(\phi)$ that usually includes the mass and interaction terms. The field always tends to the minimum of that potential in every point of physical space and when the field is in that minimum we say that it is in *the vacuum state*.
The interesting thing that can happen is that there can be more than one available vacuum, meaning that there exists more than one configuration of the field with the same minimum of the potential and no preference for the field to be in any particular one. In that case we have a "vacuum manifold" and the vacuum state can be viewed as an order parameter. One simple case is the single complex field with the following Lagrangian density:

$$\mathcal{L} = \nabla_\mu \phi \nabla^\mu \phi^* - V = \nabla_\mu \phi \nabla^\mu \phi^* - \frac{\lambda}{4}\left(|\phi|^2 - v^2\right)^2. \tag{4.1}$$

The potential is shown in figure 7 [6] and the vacuum manifold is the circle of points for which $|\phi|^2 = v^2$. One can view this as a broken symmetry, the same as in the ordered media case. For example, in the case of the field described by (4.1) the full internal symmetry group is obviously $U(1)$ and the isotropy subgroup for a particular vacuum state is just the identity since all other "rotations" change the vacuum state. Thus, as before, the vacuum manifold is $U(1)/1$ and it's topological properties are the same as for planar spins. The quantum field can, of course, have more

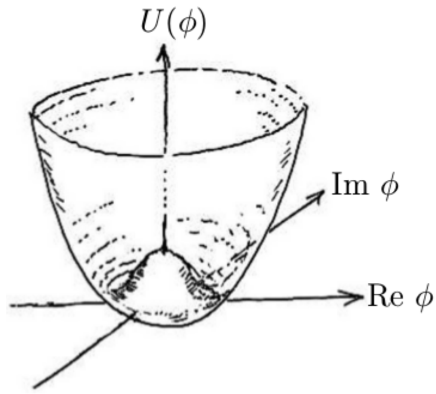

Figure 7: A "Mexican hat" potential for a complex quantum field resulting in a manifold of vacua

internal degrees of freedom with complicated symmetry groups and potentials that break them into highly non-trivial vacuum manifolds. This opens the possibility for complicated topological defects not readily accessible by intuition and of possible great importance for elementary particle physics and cosmology. Homotopy theory tells us only about the possibilities of topological defects, there are other conditions that affect their creation or even the possibility of their existence. In fact, energy considerations tell us that the existence of topological defects is impossible for theories with Lagrangians of the form like (4.1) in more than one spatial dimension (Derrick's theorem). What "saves" our standard quantum field the-

[6]This figure is taken from [6] which is an introduction to and an overview of QFT I highly recommend. It contains even short chapters on topology and its applications to QFT like anyons, Chern-Simons theory, 't Hooft-Polyakov monopole and instantons.

ories from Derrick's theorem is local gauge invariance instead of global symmetries. The shift to local gauge symmetry groups changes nothing as far as homotopy is concerned which means that our standard quantum field theories admit topological defects, at least in principle. Another question is the creation and energy content of eventual topological defects in the universe and this is were cosmology comes into play. The theory of the hot big bang says that the universe was once very hot and dense and that it expanded and cooled ever since. It also says that when it was very hot that the internal symmetries of the quantum field were not broken and that as it cooled the universe underwent a second-order phase transition that broke the internal symmetries of the quantum field. For example, the standard Landau theory would expand the potential of the (4.1) Lagrangian with a temperature dependent part $CT^2|\phi|^2 + \dots$ [7] making it:

$$V(\phi) = \left(CT^2 - \frac{\lambda v^2}{2}\right)|\phi|^2 + \frac{\lambda}{4}|\phi|^4 + const.$$

This would give a critical temperature of $T_c^2 = \frac{\lambda v^2}{2C}$ over which there would exist only a single vacuum at $\phi = 0$ and under which there would exist a vacuum manifold for $|\phi|^2 = v^2\left(1 - \frac{T^2}{T_c^2}\right) \to v^2$. This approach can be easily transferred to quantum fields with a lot more complicated internal structure and gauge symmetries, such as those of the standard model.

The way that topological defects would occur in case of such a scenario is described by *the Kibble mechanism*. At the point of the phase transition of the universe causal effects between different areas of the universe are limited by the speed of light and the age of the universe. The length scale is that of the inverse Hubble constant so we can roughly put that $l_{cor} \sim H^{-1}$. This means that areas further apart than l_{cor} couldn't have had causal effect on each other and have no reason to have the same vacuum, since all vacua are equally likely. This means that the spontaneous symmetry breaking resulted in different vacua around the universe that, due to energy considerations, settled into the uniform configuration locally and expanded. But since all vacua are equally likely and uncorrelated it should have happened that some topologically non-trivial configurations occurred that can't be settled to the uniform one. This is schematically depicted in figure 8.

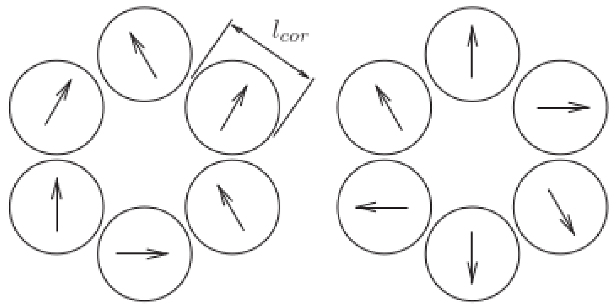

Figure 8: The Kibble mechanism leading to a topologically uniform configuration (left) and a vortex defect (right) [8]

The standard model of physics (SM) is a quantum field theory with a quantum field that has two complex components and $(SU(3)\times)SU(2) \times U(1)$ as its internal gauge symmetry group. The theory says that the field goes through spontaneous symmetry breaking that breaks the $SU_L(2) \times U_Y(1)$ gauge symmetry of the electroweak force to the $U_{em}(1)$ gauge symmetry of the electromagnetic force through the Higgs mechanism giving masses to the $W^\pm$, Z^0 and H^0 bosons. The resulting vacuum manifold cannot be calculated as easily as $(SU(2) \times U(1))/U(1) \simeq SU(2)$ because it depends on the embedding of $U_{em}(1)$ into the higher symmetry space, which you may know is not trivial (Weinberg angle etc.). However, it can be shown [9] that the resulting vacuum manifold indeed is $SU(2)$ in the theory of the Higgs $SU(2)$ doublet field. Since the group manifold of $SU(2)$ is isomorphic to S^3 it follows that the homotopy groups for the vacuum manifold $\mathcal{M}$ of SM are:

$$\pi_1(\mathcal{M}) \simeq \{e\} \quad (4.2)$$

$$\pi_2(\mathcal{M}) \simeq \{e\} \quad (4.3)$$

$$\pi_3(\mathcal{M}) \simeq \mathbb{Z} \quad (4.4)$$

This means that the SM doesn't admit cosmic strings or monopoles. However, it is postulated that there exists a more fundamental Grand Unification Theory (GUT) with a higher gauge symmetry group (for example $SU(5)$) that breaks down to the SM one in a similar way to which the electroweak symmetry breaks to the electromagnetic. Most of those GUT's admit monopoles as stable configurations, which is appealing and desired for some reasons but also generates some additional problems.

4.1 ’t Hooft - Polyakov monopole

The simplest gauge quantum field theory that admits monopoles is the Georgi-Glashow model with a quantum field having three real components and a local $SU(2)$ gauge symmetry. Its Lagrangian density is given by [6][8][10]:

$$\mathcal{L} = \frac{1}{2}(D_\mu \phi^a)(D^\mu \phi_a) - \frac{\lambda}{4}\left(\phi^2 - v^2\right)^2 - \frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a.$$

The covariant derivative D_μ and the gauge field tensor $G^a_{\mu\nu}$ are given by the three gauge fields W^a_μ and the coupling constant q through the following relations:

$$\begin{aligned} D_\mu &= \partial_\mu + q\epsilon^{abc}W^a_\mu, \\ G^a_{\mu\nu} &= \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + q\epsilon^{abc}W^b_\mu W^c_\nu. \end{aligned}$$

The vacuum state is clearly achieved for $\phi^2 = v^2$ which, considering that ϕ has three real components, makes the vacuum manifold $\mathcal{M} \simeq S^2$. This can also be seen by breaking the symmetry choosing arbitrarily the vacuum corresponding to:

$$\phi = \begin{pmatrix} 0 \\ 0 \\ v + h(x) \end{pmatrix} \tag{4.5}$$

Following the calculations through shows that gauge field combinations $W^\pm_\mu = \frac{1}{\sqrt{2}}(W^1_\mu \pm \imath W^2_\mu)$ acquire mass $m_W = qv$, that the Higgs field h acquires mass $m_h = \sqrt{2\lambda}v$ and that the gauge field W^3_μ remains massless and gives the theory a $U(1)$ gauge symmetry that can be identified with electromagnetism by $W^3_\mu \equiv A_\mu$. This means that the vacuum manifold should be $\mathcal{M} \simeq SO(3)/U(1) \simeq S^2$ which is consistent with our first conclusion. As we already know, $\pi_2(S^2) = \mathbb{Z}$ so this theory admits monopoles, an infinite variety of them even.

The simplest is the ’t Hooft-Polyakov monopole which belongs to the class with winding number 1 and is also called “the hedgehog solution” since the phase of the vacuum points in the radial direction at each point of space far from the monopole. The form of the quantum field for such a solution is not hard to guess, it is just:

$$\phi = v\,(1 - f(r))\,\hat{r}$$

where $f(r \to \infty) = 0$ and $(1 - f(r \to 0)) \propto r$. The interesting thing about this solution is that it links spatial and internal dimensions, because $\hat{r}$ “lives” in the physical space and ϕ “lives” in the three-dimensional internal space and those two spaces are generally unrelated.

The bigger problem and the reason why finding this solution was not trivial is finding the gauge fields that keep this solution finite in energy. They correspond to:

$$\begin{aligned} W^a_0 &= 0 \\ W^a_i &= \frac{1 - a(r)}{qr}\epsilon^{aij}\hat{r}^j \end{aligned}$$

where $a(r \to \infty) = 0$ and $(1 - a(r \to 0)) \propto r^2$.

In the case of a uniform vacuum chosen in (4.5) one can check to see that the electromagnetic tensor is given by the third component of the field tensor and that the others vanish: $F_{\mu\nu} = G^3_{\mu\nu}$. It is then easy to generalise this to any uniform orientation of the vacuum state since the physics shouldn’t depend on the choice of the vacuum:

$$F_{\mu\nu} = \frac{\phi^a}{|\phi|} \cdot G^a_{\mu\nu}.$$

Using this equation with the fields of the monopole configuration gives:

$$F_{ij} = -\frac{1}{qr^3}\epsilon^{ijk}\hat{r}^k$$

with all other elements equal to 0. This means that there is no electric field, only a magnetic field which is equal to:

$$\vec{B} = \frac{1}{q}\frac{\hat{r}}{r^2}. \tag{4.6}$$

This is clearly a field that a magnetic monopole would produce, completely analogous to the field of an electric monopole such as an electron. If we would extend the Maxwell equations with a magnetic monopole, making the second equation:

$$\nabla B = [\mu_0]\rho_m,$$

then for a monopole with magnetic charge q_m the field would be:

$$\vec{B} = \frac{[\mu_0] q_m}{4\pi} \frac{\hat{r}}{r^2}. \tag{4.7}$$

Comparing (4.7) with the monopole solution (4.6) we see that the magnetic monopole charge of the 't Hooft-Polyakov monopole (ignoring μ_0 and ϵ_0) is:

$$q_m = \frac{4\pi}{q} \to \frac{2h}{q} \tag{4.8}$$

where the second result is after restoring appropriate factors of $\hbar$. One should keep in mind that q is here the coupling constant of the electromagnetic gauge theory which is the quantum of electric charge.

The reason why this result is very interesting, and also why one would like a GUT to have monopole solutions like this, is a classical result by Dirac that says that if we extend Maxwell's equations with magnetic monopoles the magnetic charge must obey the relation:

$$q_m = n\frac{h}{q}. \tag{4.9}$$

We see that the 't Hooft-Polyakov monopole indeed obeys that relation, with $n = 2$, which is a stunning fact considering how vastly different the theories the two results were obtained with are. The derivation of the condition in (4.9) and more about the connection of the Dirac and 't Hooft-Polyakov monopoles can be found in section 10.4 of [10].

What the classical result doesn't say is anything else about the hypothetical magnetic monopole, it's just added to the Maxwell's equations by hand and all its other physical properties, like its mass, are completely arbitrary. The 't Hooft-Polyakov monopole, on the other hand, has all of its other properties determined by the theory it results from. For example, in the simplest model presented here the mass of the monopole is estimated at $m_M \approx m_W/\alpha \approx 137 m_W \approx 11$ TeV. This is around 7 orders of magnitude heavier than the electron. In GUTs, however, the monopoles usually have masses of the order of 10^{17} GeV, which is of the order of micrograms, and are formed in a phase transition that occurs at a critical temperature of roughly 10^{16} GeV. This results in "the monopole problem" which states that standard GUTs lead via the Kibble mechanism to an overabundance of monopoles, both in number and in energy density, which is completely inconsistent with current observational knowledge of our universe [8]. This "monopole problem" is one of the main reasons why the inflation mechanism, which seems to provide a way around it for some GUTs at least, was invented.

4.2 Cosmic strings

The simplest model having a vacuum manifold with $\pi_1(\mathcal{M}) \neq 0$ after symmetry breaking and so allowing 1D topological defects in 3D space is the abelian Higgs model. It is basically the Lagrangian given by (4.1) extended with a $U(1)$ local gauge field. The simplest such topological defect in that model is an infinite straight string. It can be described by the following configurations of the quantum and gauge fields:

$$\phi = v\,(1 - f(\rho))\, e^{iN\theta} \tag{4.10}$$

$$A_i = -\frac{N}{q\rho^2}\,(1 - a(\rho))\, \epsilon_{ij} r^j \;;\; i, j \in \{1, 2\} \tag{4.11}$$

where ρ is the distance from the string, the z-axis is pointing in the direction of the string and the f and a functions have appropriate asymptotic behaviour:

$$f(\rho \to \infty) = 0\;,\; f(\rho \to 0)) = 1 \quad ,$$
$$a(\rho \to \infty) = 0\;,\; a(\rho \to 0)) = 1 \quad .$$

In the center of the string the quantum field is obviously not in a vacuum state which means that the string carries energy. For the simplest abelian Higgs model used here the estimated tension (energy per unit length) of the string is $\mu \approx \pi v^2$ and its thickness is $\rho \approx 1/qv$ which is of the order of 1 fm. The size and energy of such strings make them very interesting to cosmology because of their potential implications.

An interesting property of the (4.10, 4.11) field configurations is that the string core carries a magnetic flux and that flux is quantized:

$$\int \vec{B}\mathrm{d}S = \oint A_\theta \mathrm{d}\theta = \frac{2\pi}{q} N \quad .$$

This same property is found in vortices in type II superconductors that "leak" magnetic flux when the external magnetic field exceeds a critical value. Their mathematical origin is similar and in superconductors they are called Abrikosov vortices while in cosmology they are called Abrikosov-Nielsen-Olesen strings, or simply cosmic strings.

Cosmic strings can be open, horizon-sized objects or finite-sized, closed loops and both varieties are predicted to be generated at a phase transition according to the Kibble mechanism, irrespective of the details of the theory that supports them.

Straight strings wouldn't gravitate but would affect the geometry of spacetime considerably. The geometry around them would be locally flat but with a so-called deficit angle $\Delta\theta \approx 8\pi G\mu$ that would make the full circle around the string be $2\pi - \Delta\theta$. The effects of that would be the creation of double images and non-Gaussian distortions of the CMB. The observations of the CMB thus put a bound on the tension of horizon-sized cosmic strings at $\mu \leq (0.7\cdot 10^{16}\ \mathrm{GeV})^2$. Another effect would be the compression of dust in wakes of moving cosmic strings which could have perhaps generated seeds for structure formation. Some observed 1D structures in the universe could be explained in that way but are inconsistent with some others. Generally, the hypothesis cannot be easily dismissed.

Horizon-sized strings needn't be straight and those with "ripples" and singularities like kinks and cusps would gravitate and dissipate their energy through gravitational waves which could also be detected. Additionally, such irregular strings could generate smaller closed strings by self-intersection.

Closed strings, if they existed, would gravitate and dissipate their energy through gravitational waves. As it dissipates energy a closed string would get ever smaller and when its radius would become as small as its thickness it would look like a high-energy particle. Such a particle-like topological defect would, in the simple abelian Higgs theory, have a mass similar to the monopole but would, unlike a monopole, have a completely contained magnetic flux and would interact only gravitationally making it a possible WIMP.

Although the standard model doesn't allow cosmic strings various GUTs might and different varieties of cosmic strings, but with very similar properties, are allowed in various string and superstring theories. This, coupled with the numerous possible fundamental consequences the existence of cosmic strings would have for our universe, makes them a very interesting topic of study. More details about the various cosmological consequences of cosmic strings, along with some calculations, can be found in [8].

4.3 Kinks and walls

The breaking of symmetry can, in principle, also lead to a disconnected vacuum manifold, a manifold with $\pi_0(\mathcal{M}) \neq 0$. Such a theory would allow for the existence of topological defects called domain walls which would present a boundary between two parts of space with different vacua. The most simple example of such a theory is a single scalar field ϕ with the following Lagrangian:

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{\lambda}{4}\left(v^2 - \phi^2\right)^2 \qquad (4.12)$$

which obviously has two possible vacua: $\phi(x) = \pm v$. If we consider only one spatial dimension a domain wall is usually called *a kink*, the reason for which is clear when looking at figure 9. A static kink solution can be easily constructed by any function with the desired asymptotic behaviour, for example:

$$\phi(x) = v \tanh\left(\frac{x}{l}\right) \quad ; \quad l^2 = \frac{2}{\lambda v^2}. \qquad (4.13)$$

Since the field is not everywhere in a vacuum state a kink carries energy, but that energy is localized in a region of width l. The energy is easily calculated and

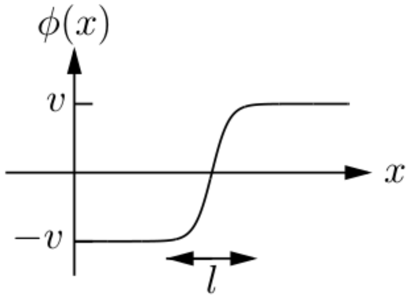

Figure 9: A "kink" - a domain wall in one dimension

is finite. The given solution can be trivially extended to two and three spatial dimensions which would presume a straight line or a flat plane as the boundary between domains. This, of course, doesn't have to be the case but if the wall is not strongly curved it is a good approximation.

The cosmological consequences of horizon-sized domain walls would be considerable and certain conditions can be constructed from existing observations. Simple energy density considerations already give an unlikely satisfiable boundary of surface energy density $\eta \leq (10\mathrm{GeV})^3$ given the usual symmetry breaking energy scales for standard model extensions. An additional likely effect of a domain wall would be that it anti-gravitates which would have a big effect on CMB. The CMB observations thus put an even tighter upper bound on the surface energy density of a horizon-sized domain wall. These results are the reason why the topological possibility of domain walls is undesired in theories that extend the standard model.

4.4 Textures and other wild beasts

The same as in ordered media, textures are possible, defectless configurations of the vacuum if the vacuum manifold has a non-trivial third homotopy group. In quantum field theories of the kind we discussed so far they are also created by the Kibble mechanism during a symmetry breaking phase transition and they are energetically unstable. In broken gauge theories textures relax by emitting Nambu-Goldstone bosons and they decay too fast to still be present. Their effect could be indirectly seen through density perturbations in the early universe reflected in the CMB but so far observations are inconsistent with textures being a relevant factor [8].

Another kind of a topological defect that depends on the third homotopy group is *an instanton*. Instantons are solutions that are localised both in space and time and they are studied on the four-dimensional Euclidean space (after Wick-rotating the time dimension) which is why the third homotopy group is relevant and the vacuum manifold can facilitate configurations with defects.
Instantons are too complicated, both mathematically and physically, to go into any more depth here. They are mentioned only to point out to the reader that they are possible because of non-trivial topology of the vacuum manifold making them also topological defects. A very elementary introduction to instantons with some intuitive explanations and elementary calculations can be found in the last chapter of [6] and a bit more mathematically involved introduction can be found in section 10.5 of [10].

The last kind of topological defects that I will mention in this essay are the so-called *hybrid topological defects*. They are combinations of defects described so far and are hypothetically possible if a theory went through multiple symmetry breaking phase transitions where different kinds of allowed defects were produced at each transition. Examples of such exotic topological structures are *fleece* which are cosmic strings that end on domain walls, and *necklaces* which are cosmic strings containing monopoles [8].

5 Conclusion

Physicists dealing with condensed matter have long ago realised the possibility of defects in materials and managed to describe them and deal with them without using formal mathematics from topology and homotopy theory. Modern physics has, however, given us theories that are both mathematically and in the space, time and energy scales they describe far from the possible grasp of our intuition. It is that circumstance that brought topology and homotopy theory to condensed matter physics, but they have since proven indispensable for the accurate and systematic explanation and prediction of the existence and the dynamics of topological defects in even classical ordered media.
We have also seen that quantum field theory often gives topologically rich vacuum manifolds. Homotopy theory has here also proven indispensable and has revealed the possibility of a variety of topological defects with interesting properties existing in our universe. The systematic exploration of those possibilities using methods from topology can tell us what to look for to confirm the theories we have but also what we should expect of the theories that are candidates for their successors.
Homotopy theory and topology in general are obviously a necessary tool for modern physics if all implications of the mathematically complicated physical theories are to be understood and especially if those theories are to be meaningfully applied to the totality of our universe.

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